Indirect Proofs

Announcements

- ***Pset 0***
  - Due Monday (but try to finish by today)
- ***Pset 1***
  - Goes out today, due next Friday
  - A LaTeX Beginner's Quick Start Tutorial video from Winter has been posted to Canvas under Panopto Course Videos (LaTeX is the preferred tool for writing homework in this class)
  - Partners are allowed—go to Ed Q&A forum to find one
- ***Office Hours***
  - They start Monday! Schedule of Zoom meetings will be on the course website later today.

Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Logical Implication

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the ***antecedent***.

This part of the implication is called the ***consequent***.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

What Implications Mean

- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.
- Direction matters—this does not say that any time Q(x) is true, P(x) will be true!
- The statement only says or means anything when P(x) is true, otherwise “all bets are off.”

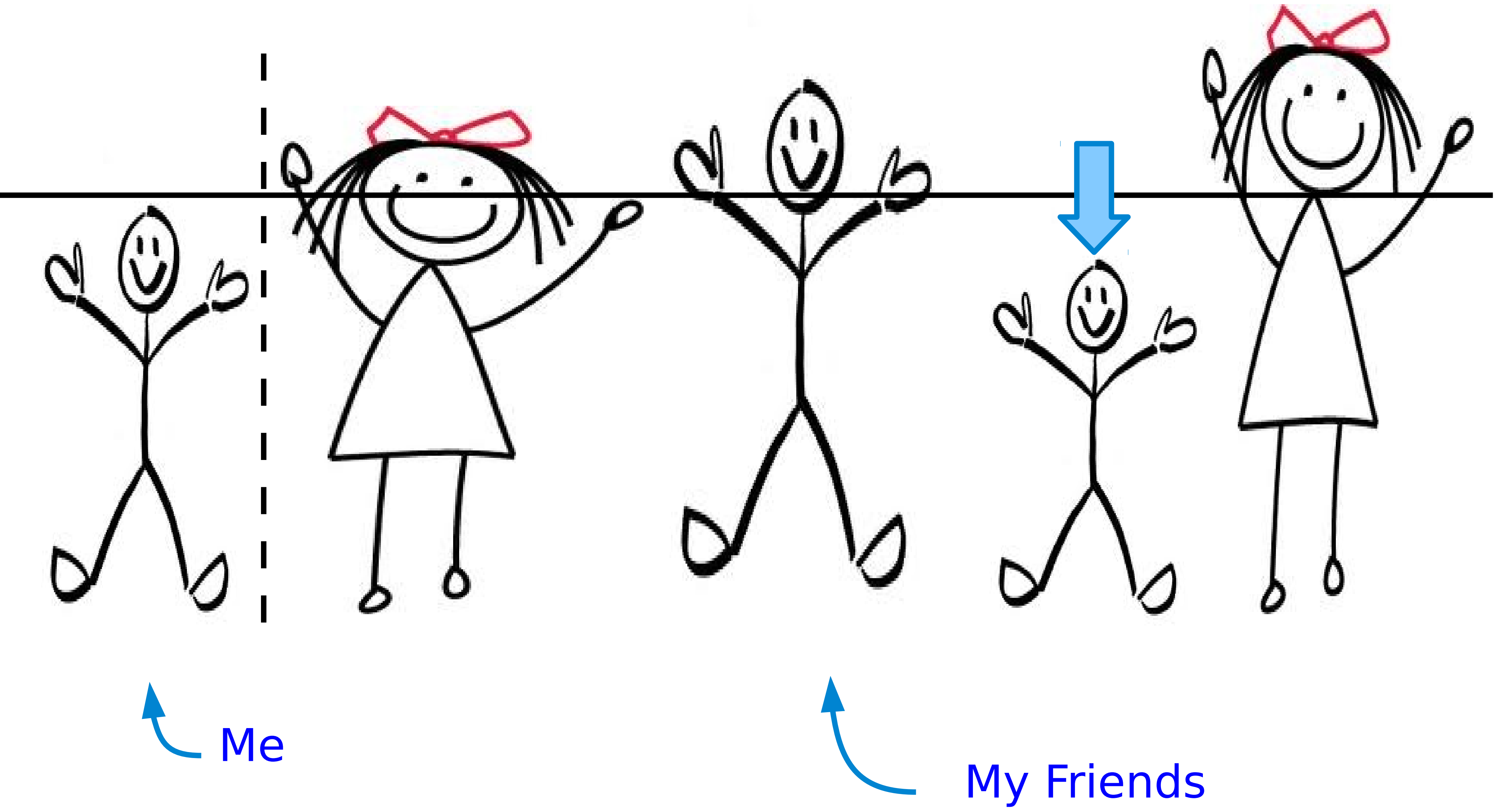
Negations

Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the proposition "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠
  - Its negation is *not* "we're in the Bay Area." ⚠

How do you find the negation of a statement?

“All My Friends Are Taller Than Me”



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

(Remember that existential means "at least one.")

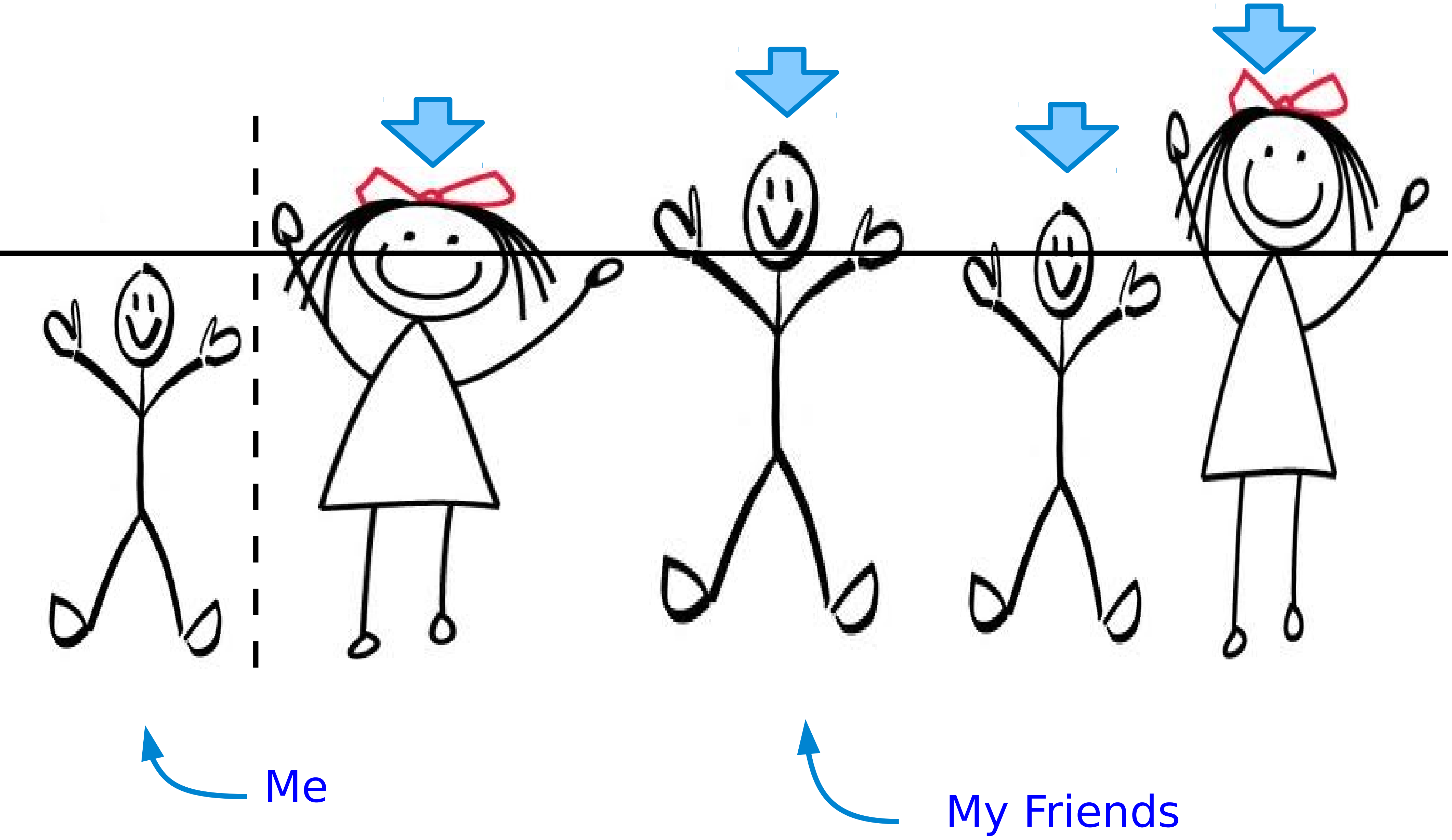
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the ***universal*** statement

For all *x*, *P*(*x*) is false.

How do you negate an implication?

Negating Implication

Dr. Lee: "***If*** you pick a perfect March Madness bracket this year, ***then*** I'll give you an A+ in CS103."

Q: under what conditions am I a liar?*
What if…

- …you pick a perfect bracket and get an A+?
- …you pick a bad bracket and get an A+?
- …you pick a perfect bracket and get a C?
- …you pick a bad bracket and get a C?

** The way we define negation in logic means that the negation is true only under the conditions in which I am a liar.*

The negation of the statement

“For any *x*, if *P*(*x*) is true,
then *Q*(*x*) is true”

is the statement

“There is at least one *x* where
***P*(*x*) is true and *Q*(*x*) is false.”**

The negation of an implication
is not an implication!

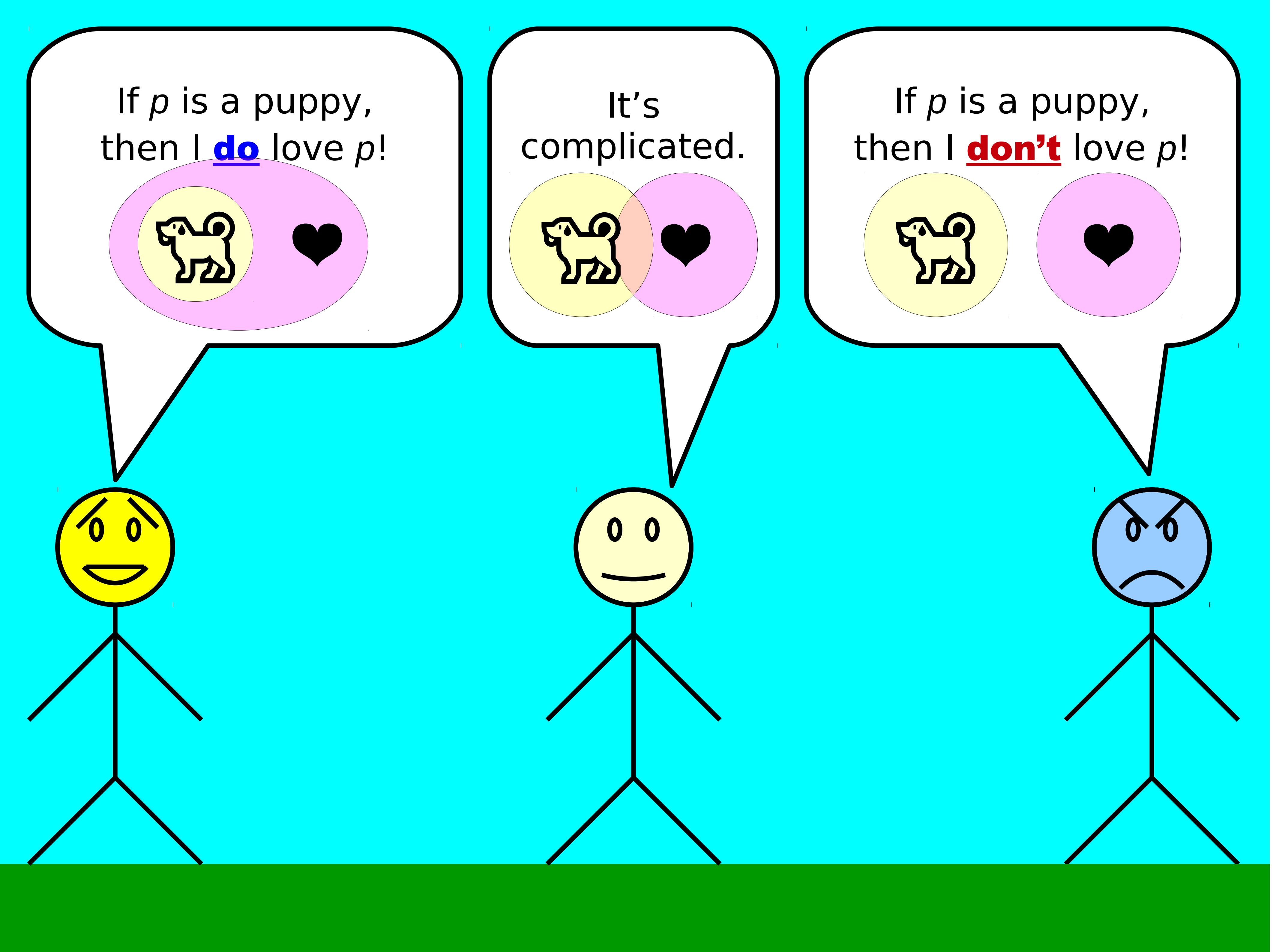
The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!


If *p* is a puppy, then I **do** love *p*!
It's complicated.
If *p* is a puppy, then I **don't** love *p*!

How to Negate Universal Statements:

"For all *x*, *P*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is false."

How to Negate Existential Statements:

"There exists an *x* where *P*(*x*) is true"

becomes

"For all *x*, *P*(*x*) is false."

How to Negate Implications:

"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

becomes

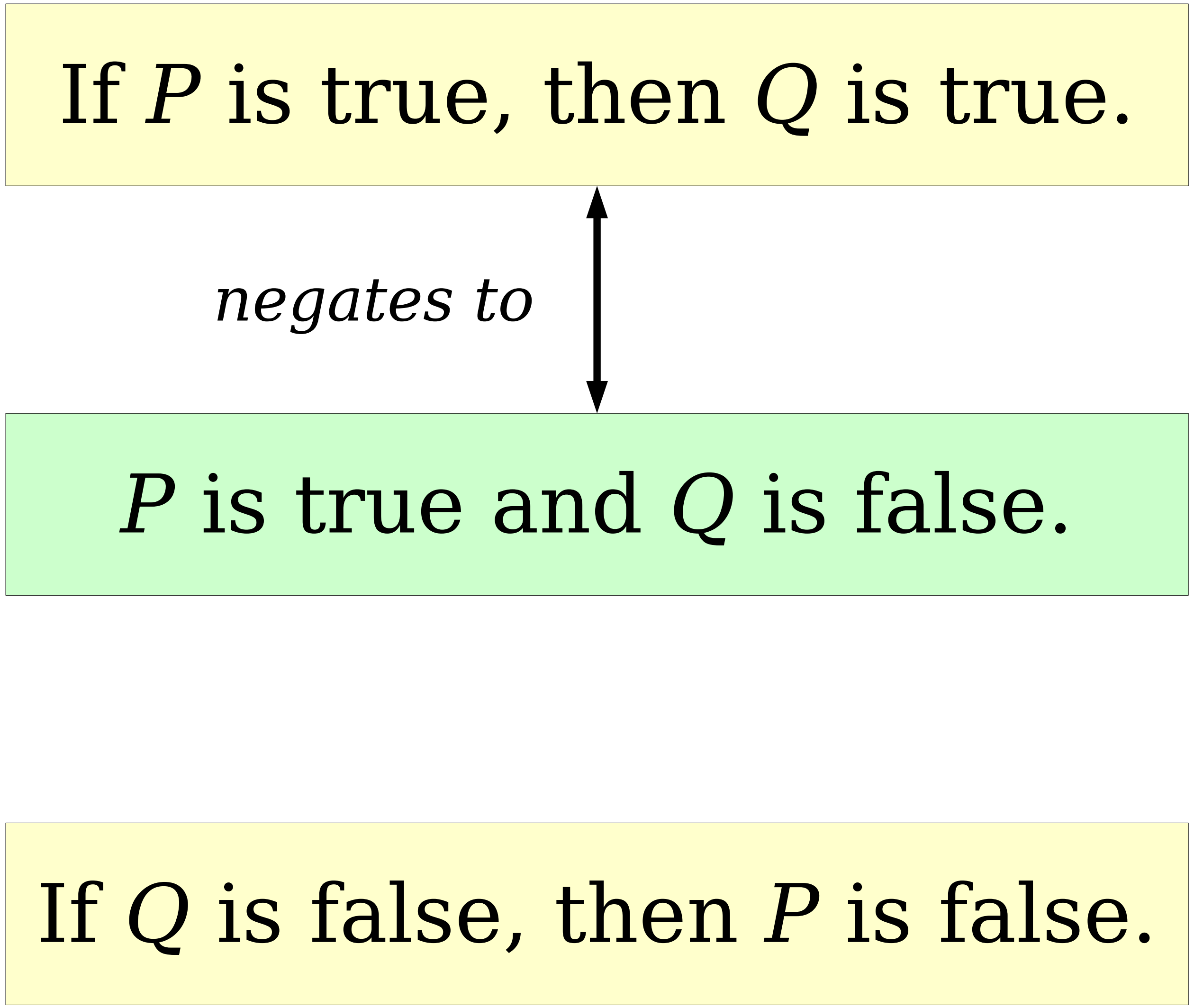
"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false."

Proof by Contrapositive

If P is true, then Q is true.

If Q is false, then P is false.

What are the negations of the above two statements?



What are the negations of the above two statements?

If P is true, then Q is true.

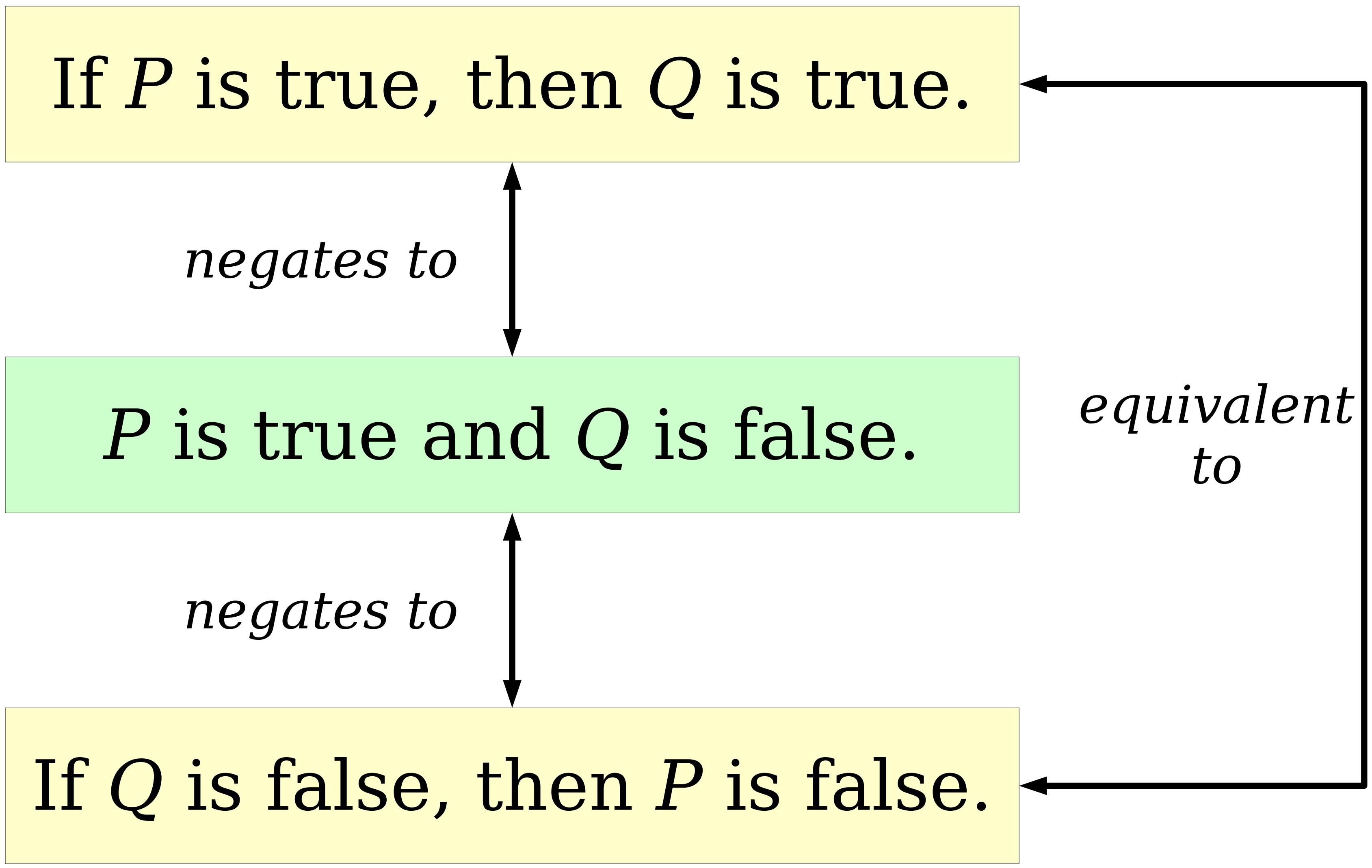
negates to

P is true and Q is false.

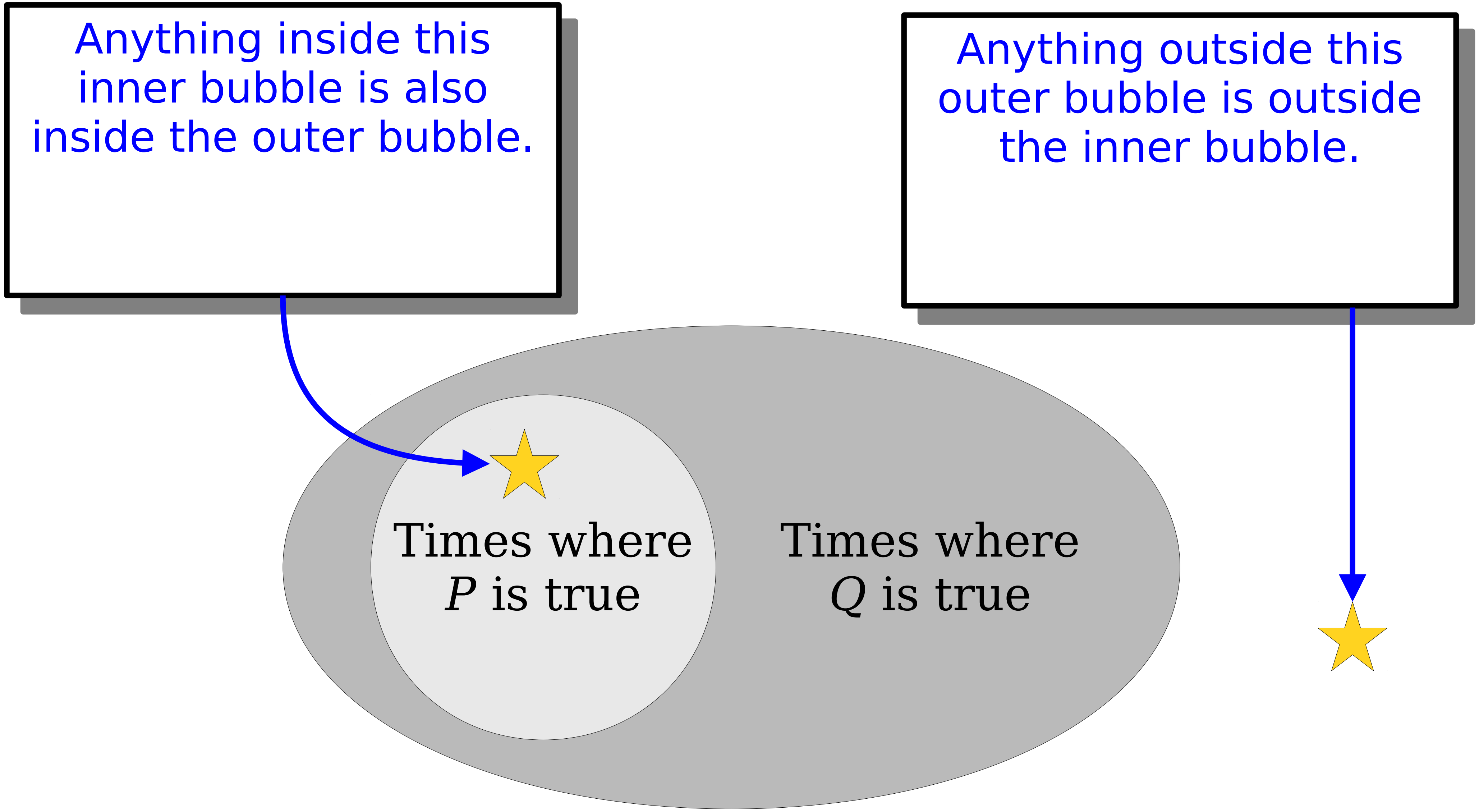
negates to

If Q is false, then P is false.

What are the negations of the above two statements?



What are the negations of the above two statements?



If P is true, then Q is true.

If Q is false, then P is false.

The Contrapositive

- The ***contrapositive*** of the implication

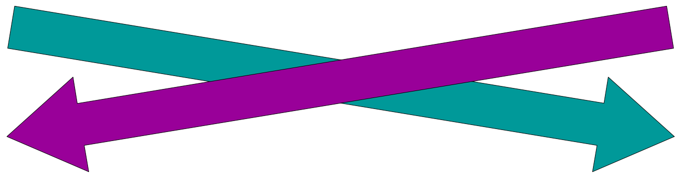
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

The Contrapositive

- The ***contrapositive*** of the implication

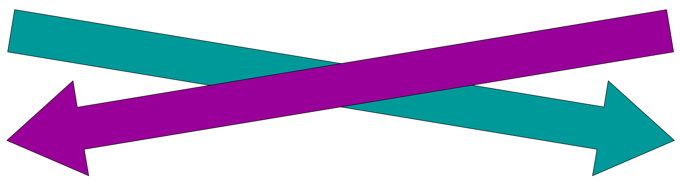
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won't steal it.



If raccoons stole the cat food, then I didn't store it inside.

To prove the statement

“if *P* is true, then *Q* is true,”

you can choose to instead prove the equivalent statement

“if *Q* is false, then *P* is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

This is a courtesy to the reader and says “heads up! we’re not going to do a regular old-fashioned direct proof here.”

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

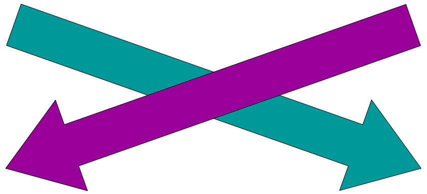
if n^2 is even, then n is even.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.



Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1
\end{aligned}
$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd.

We know ...
integer ...
us that ...

From th...
(namely ...
means t...
to show. ■

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

  For any integer n, if n is even, then n^2 is even.

  For any integer n, if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

  for any integer n: n is even if and only if n^2 is even.

Proving Biconditionals

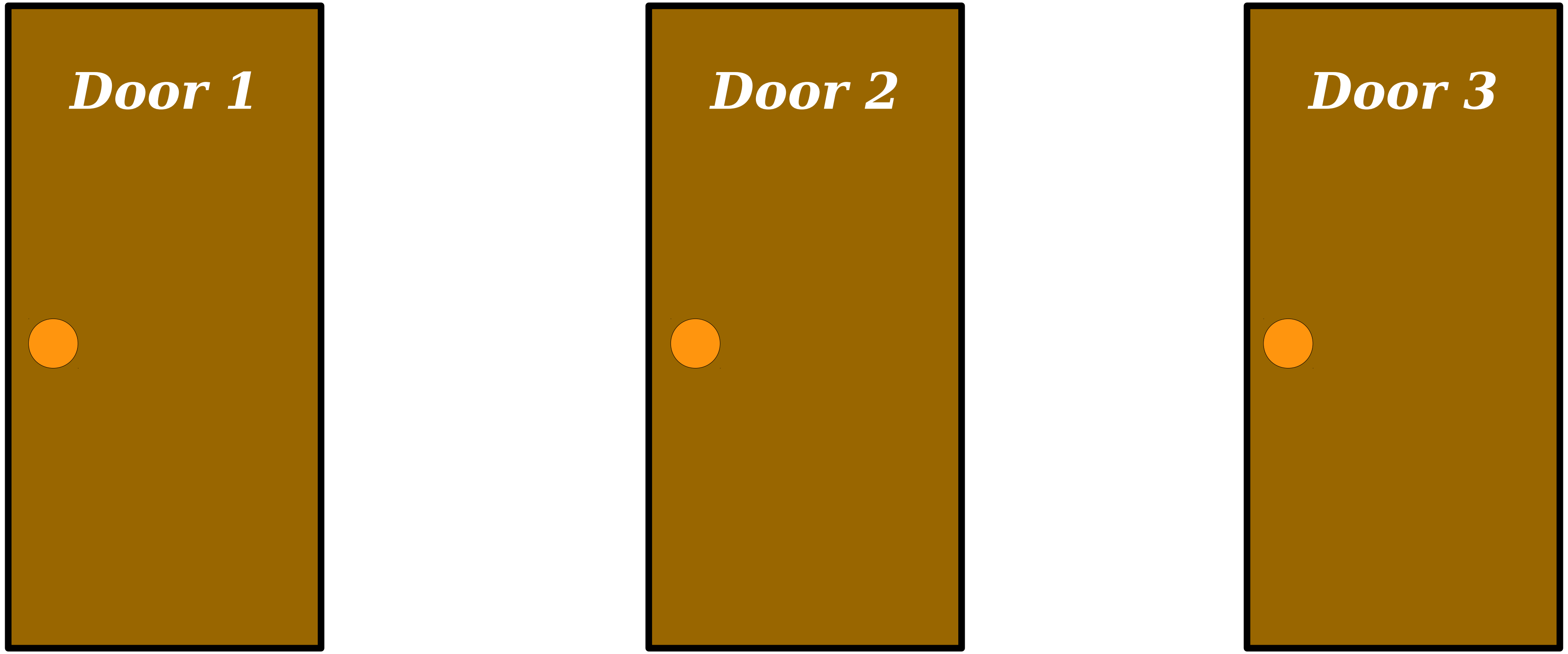
- To prove a theorem of the form

  ***P* if and only if *Q*,**

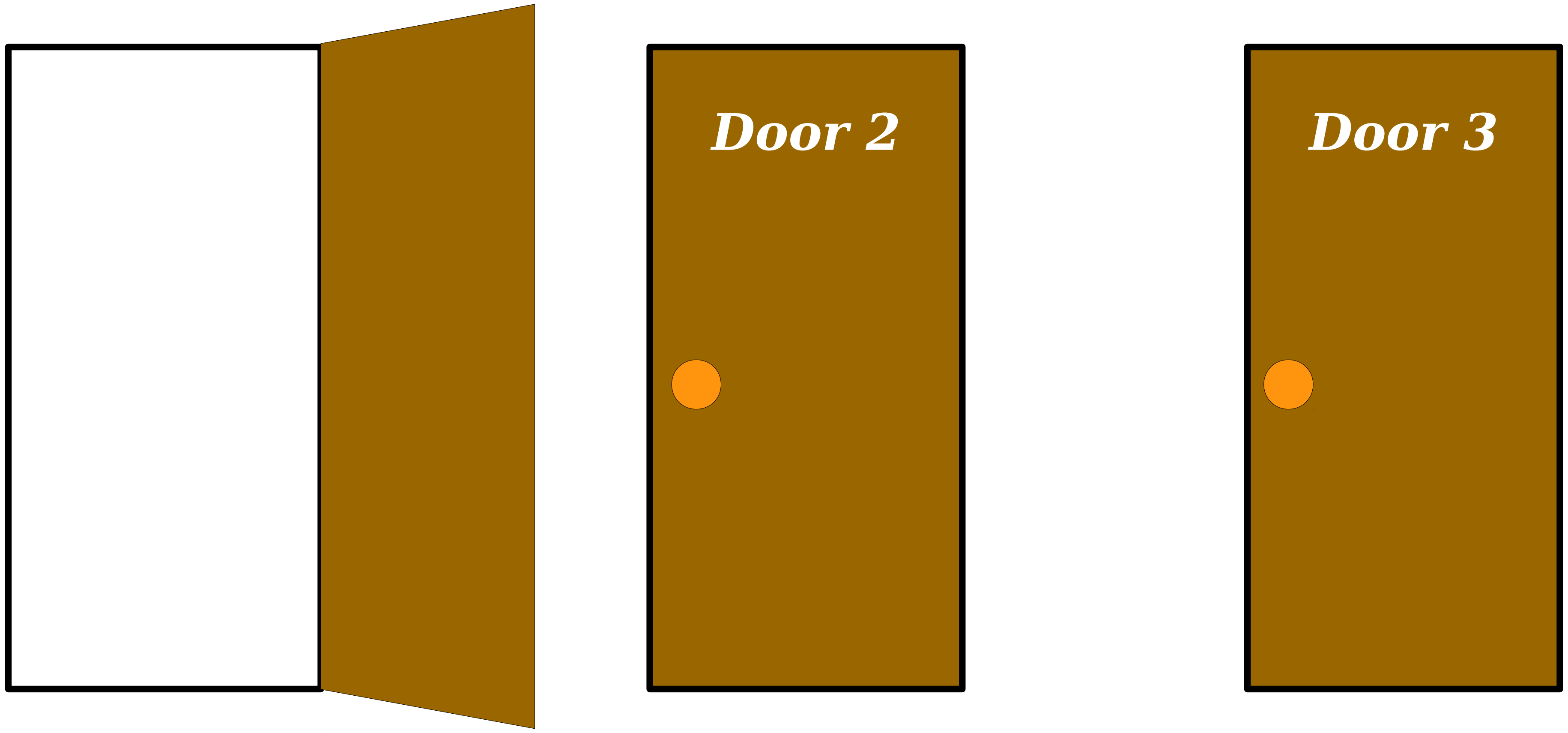
  you need to prove two separate statements.
  - First, that if P is true, then Q is true.
  - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Proof by Contradiction

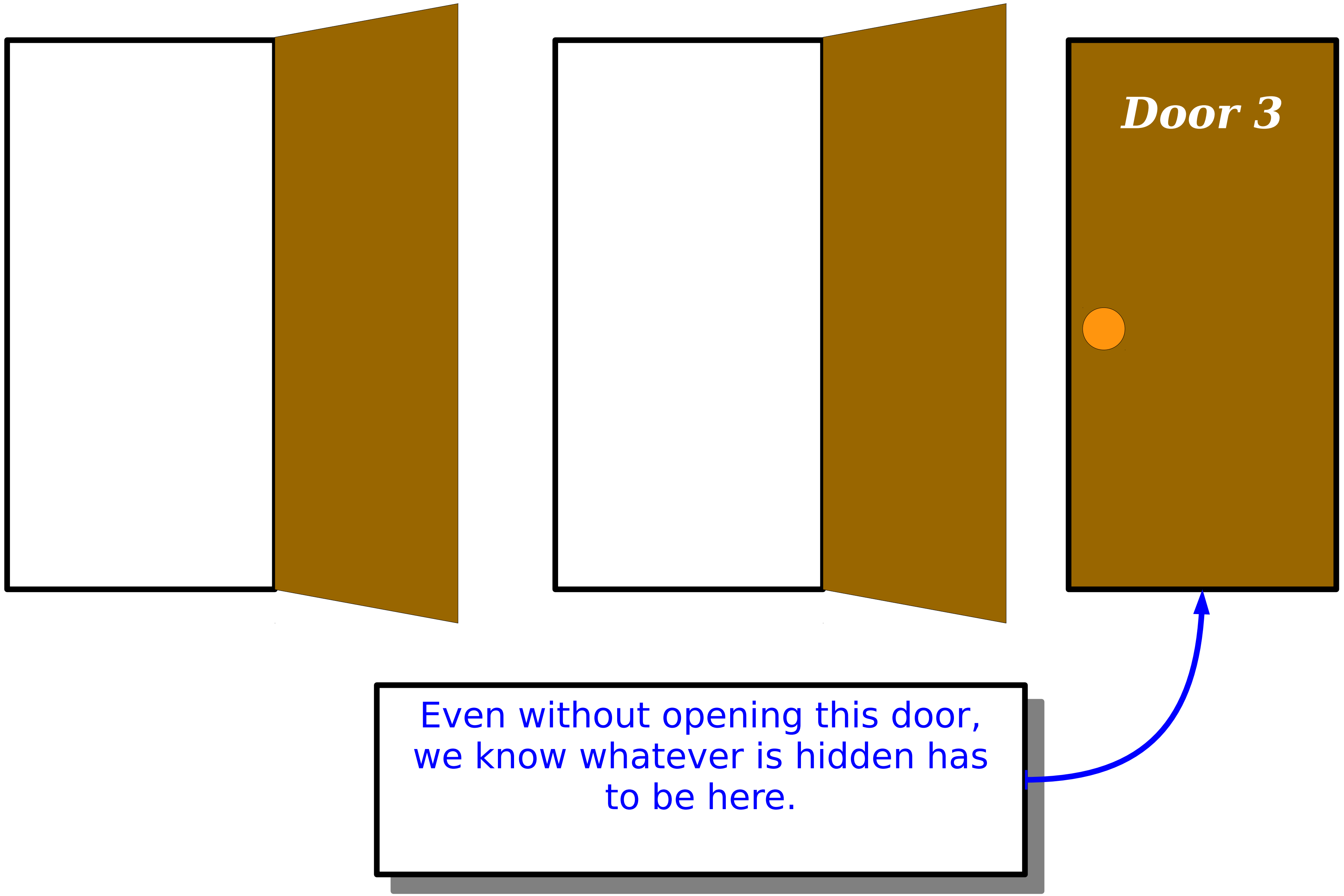
There's something hidden behind one of these doors.
Which door is it hidden behind?



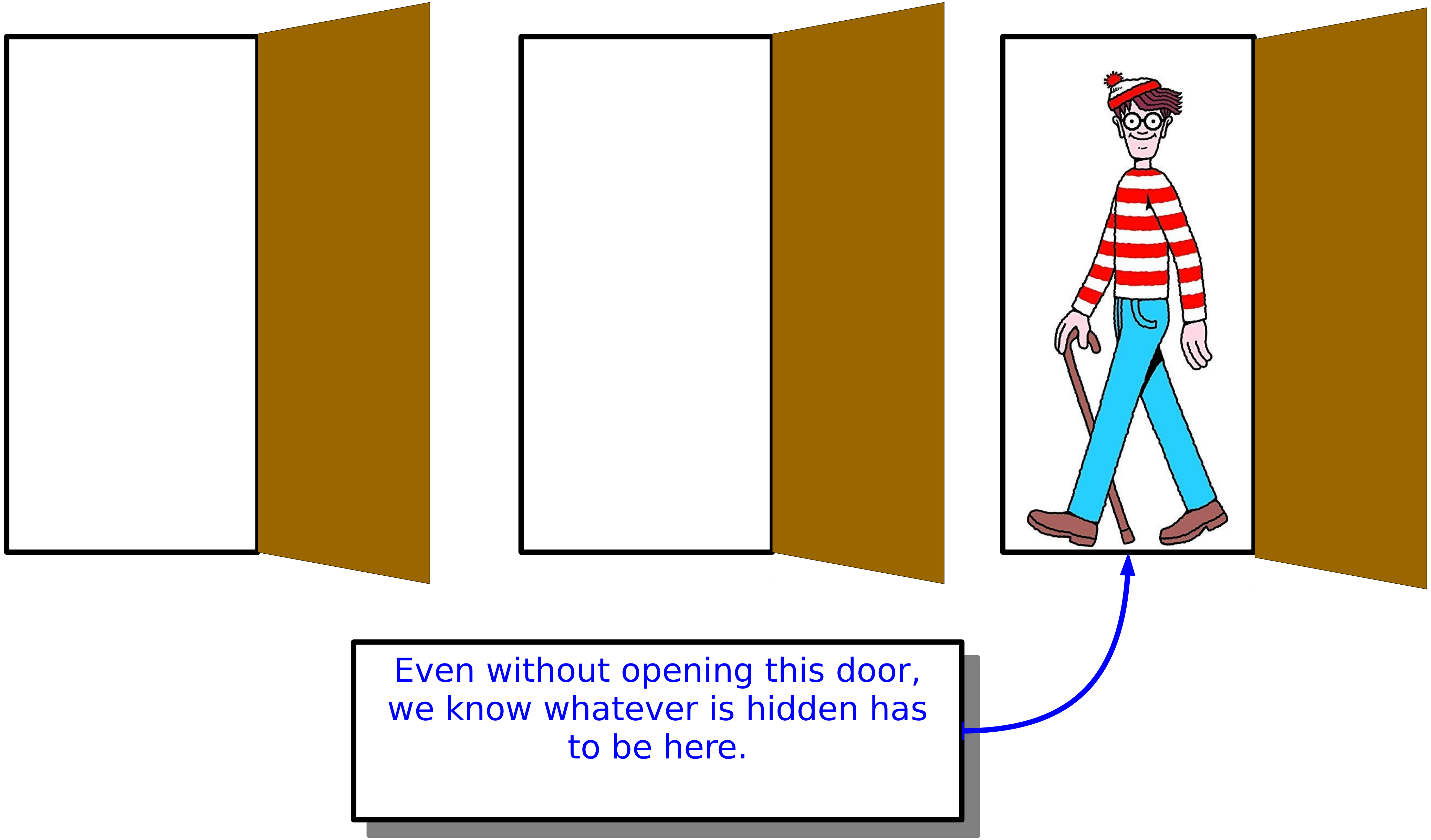
**There’s something hidden behind one of these doors.
Which door is it hidden behind?**



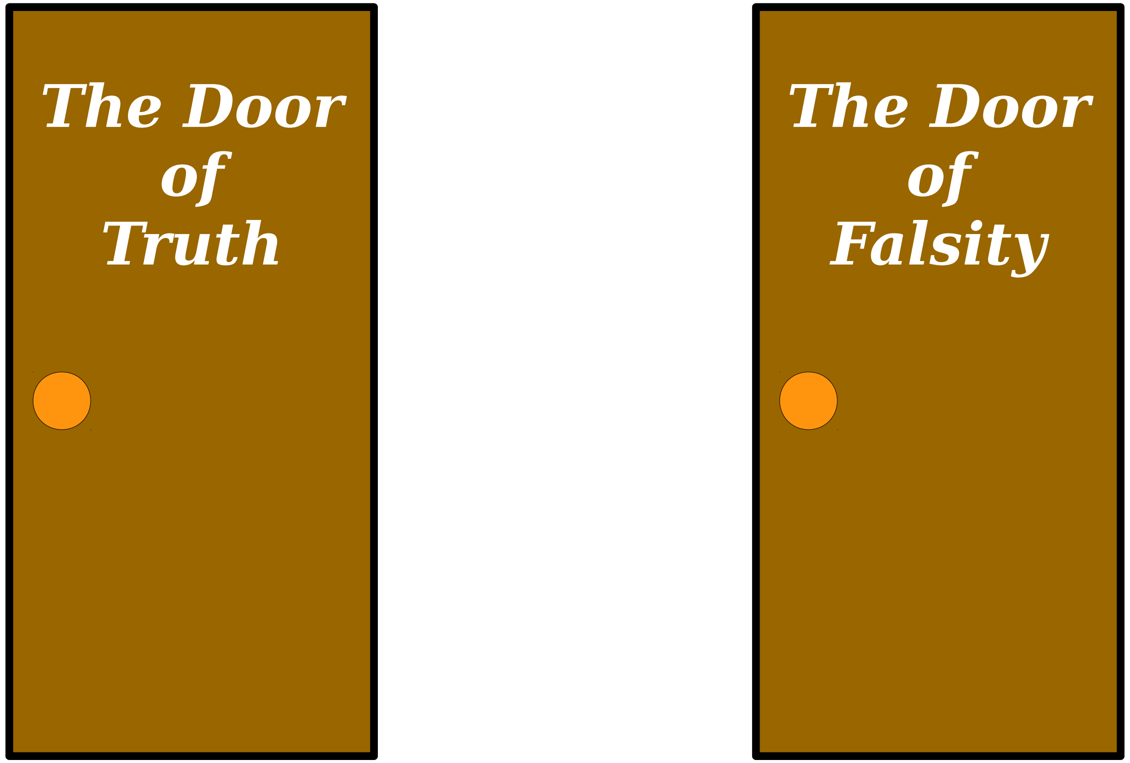
There's something hidden behind one of these doors.
Which door is it hidden behind?



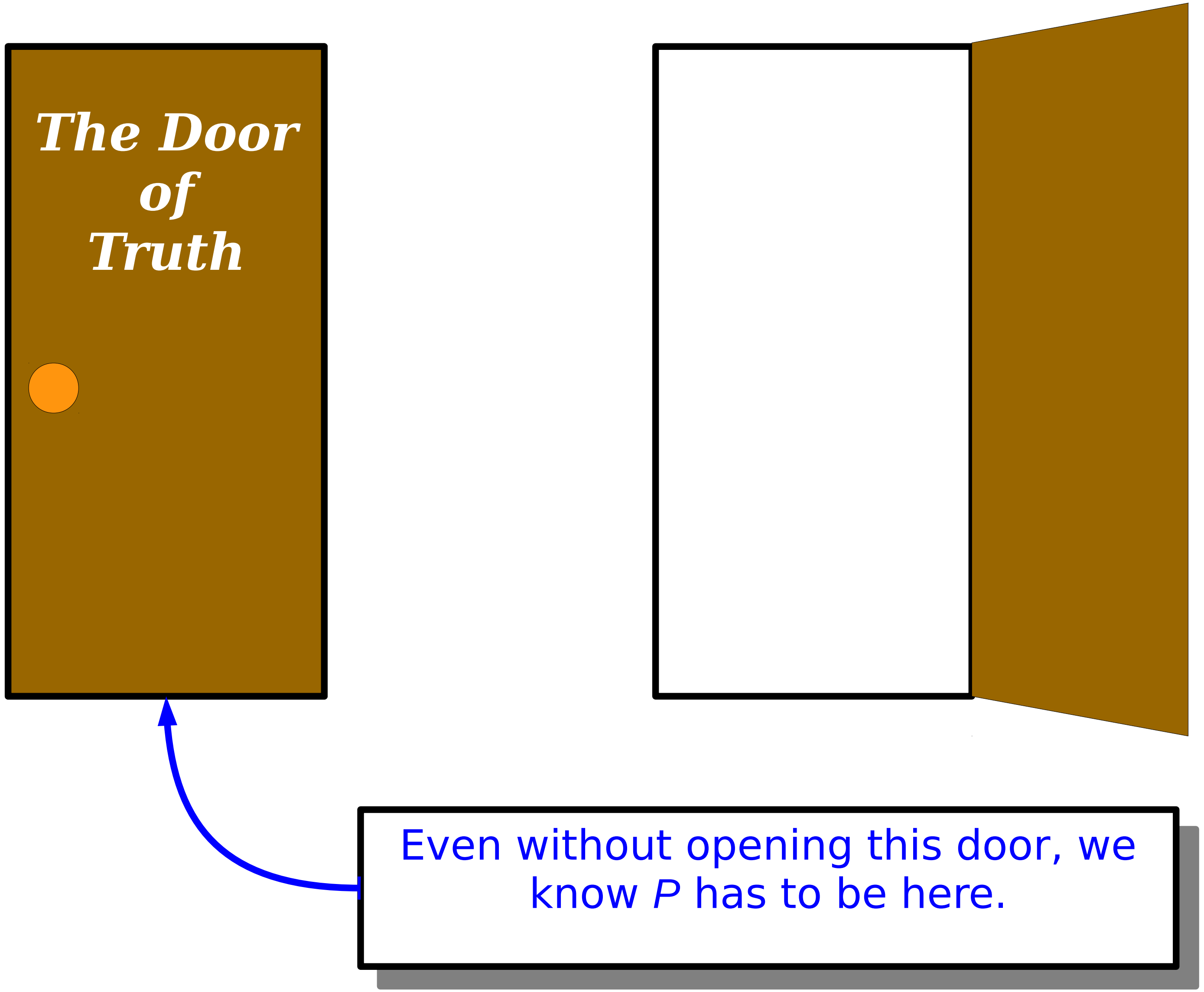
There's something hidden behind one of these doors.
Which door is it hidden behind?



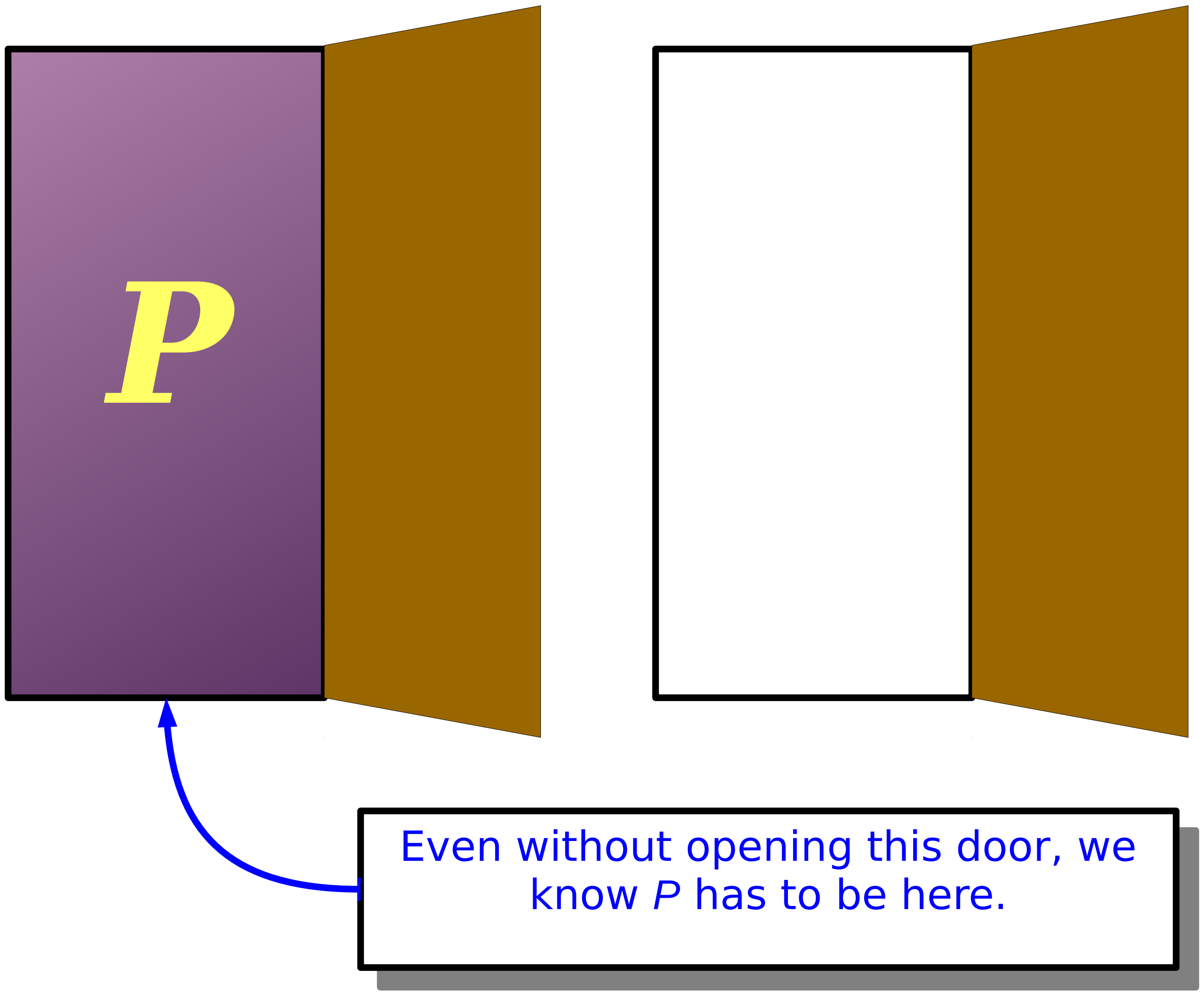
Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?



Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?



Every statement in mathematics is either true or false.
If statement P is **not false***, what does that tell you?***



A ***proof by contradiction*** shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

Theorem: There is no largest set.

Proof:

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Notice that we're announcing

1. **that this is a proof by contradiction**, **and**
2. **what, specifically, we're assuming**.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Proving Implications

- Suppose we want to prove this implication:

If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***
  - ***Proof by Contrapositive.***
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P*** **is true**, then prove ***Q*** **is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q*** **is false**, then prove that ***P*** **is false**.
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

    … what does this look like?

Theorem: For any integer n, if n^2 is even, then n is even.

Theorem: For any integer n, if n^2 is even, then n is even.

What is the negation of our theorem?

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Theorem: For any integer n, if n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \qquad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

Theorem: For any integer n, if n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

Theorem: For any integer n, if n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect.

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \qquad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \qquad (2) \end{aligned}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well.

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \qquad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \qquad (2) \end{aligned}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such

The three key pieces:

1. **Say that the proof is by contradiction.**
2. **Say what the negation of the original statement is.**
3. **Say you have reached a contradiction and what the contradiction entails.**

In CS103, please include all these steps in your proofs!

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Theorem: For any integer n, if n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.

  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.

  - ***Proof by Contradiction.***

    Assume ***P is true*** and ***Q is false***,
    then derive a contradiction.

What We Learned

- ***What's an implication?***
  - It's statement of the form “if P, then Q,” and states that if P is true, then Q is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of “if P, then Q” is “if not Q, then not P.”)
- ***What's a proof by contradiction?***
  - It's a proof of a statement P that works by showing that P cannot be false.

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.